

Classification of the classical $SL(2, \mathbb{R})$ gauge transformations in the rigid body

Manuel de la Cruz,^{*} Néstor Gaspar,[†] Lidia Jiménez,[‡] and Román Linares[§]
Departamento de Física, Universidad Autónoma Metropolitana Iztapalapa,
San Rafael Atlixco 186, C.P. 09340, México D.F., México,

In this paper we revisit the classification of the gauge transformations in the Euler top system using the generalized classical Hamiltonian dynamics of Nambu. In this framework the Euler equations of motion are bi-Hamiltonian and $SL(2, \mathbb{R})$ linear combinations of the two Hamiltonians leave the equations of motion invariant, although belonging to inequivalent Lie-Poisson structures. Here we give the explicit form of the Hamiltonian vector fields associated to the components of the angular momentum for every single Lie-Poisson structure including both the asymmetric rigid bodies and its symmetric limits. We also give a detailed classification of the different Lie-Poisson structures recovering all the ones reported previously in the literature.

I. INTRODUCTION

Taking the Liouville theorem as a guiding principle, in [1] Yoichiro Nambu proposed a generalization of the classical Hamiltonian dynamics by supersede the usual two dimensional phase space with an n -dimensional one. The dynamics in the new phase space was formulated via an n -linear fully antisymmetric bracket (Nambu bracket) with two or more “Hamiltonians” instead of the standard Poisson bracket with one Hamiltonian. As an example Nambu applied his formalism to a well understood system in both classical and quantum mechanics, the free rigid body or Eulerian top for short. For this system the phase space is three-dimensional and is generated by the components of the angular momentum \vec{L} in the body-fixed frame, the two Hamiltonians or Casimir functions are given by the total kinetic energy E and the square of angular momentum $L^2 \equiv G$, while the Euler equations of motion are written in terms of the Nambu bracket as $\dot{L}_i = \{L_i, E, G\} = \nabla L_i \cdot (\nabla E \times \nabla G)$. In this framework the problem is manifestly invariant under linear *canonical transformations* on the triplet $\vec{L} \rightarrow \vec{L}'$ which belong to the group $SL(3, \mathbb{R})$, and under $SL(2, \mathbb{R})$ linear combinations of the constants of motion E and G , which were referred originally as *gauge transformations* by Nambu, nomenclature we use throughout this paper. It is important to note that these two sets of symmetries are not independent, in fact the gauge transformations are a subset of the canonical transformations [1]. Working explicitly with the symmetries in the Nambu classical dynamics is the base of its power. It has been recognized that Nambu brackets are the tool to describe the classical evolution of systems with a number of constants of motion beyond the minimum required for integrability of the system (see for instance [2, 3] and references therein). Geometrically the Nambu structure is related to a volume-element Jacobian determinant in the higher dimensional phase space which makes the formalism suitable to describe extended objects. This property has been exploited in the search for a formulation of M-theory (see for instance [4–6] and references therein). Not less important is the relation that Nambu brackets have with several branches of Mathematics such as Algebraic geometry and Group theory (see for instance [7] and references therein).

With the quantization of the rigid body in the three-dimensional phase space as his aim, Nambu identified the different Lie structures for the Quantum generators obtaining as result that the possibilities include the $SO(4)$ Lie algebra and the ones related to it via analytic continuation or group contraction [1]. In other words, the Lie algebras of the generators are the ones corresponding to the different subgroups of $SL(3, \mathbb{R})$. Some years later Holm and Marsden studied and classified the classical $SL(2, \mathbb{R})$ gauge symmetries [8]. Explicitly, they built new Casimir’s using $SL(2, \mathbb{R})$ combinations of the original ones E and G , concluding that the Hamiltonian form of these combinations can be written as a Lie-Poisson system associated to a Lie algebra structure on \mathbb{R}^3 and establishing that the different Lie algebras are $SO(3)$, $SO(2, 1)$, $ISO(2)$ and Heis_3 . Particularly interesting is the $ISO(2)$ case where the phase space of the Eulerian top is filled with invariant elliptic cylinders on each of which the dynamics is, in elliptic coordinates, the dynamics of a standard simple pendulum. The quantization of these systems, some times referred as extended free rigid bodies, has been reviewed recently finding explicit solutions only for the symmetric bodies [9, 10]. Interesting

^{*}Electronic address: fisikito@gmail.com

[†]Electronic address: nex3t.gr@gmail.com

[‡]Electronic address: lidia@xanum.uam.mx

[§]Electronic address: lirr@xanum.uam.mx

for our purposes is that in this classification the authors also identified the algebra $ISO(1, 1)$ as a possible Lie-Poisson structure.

The aim of this contribution is to revisit the classification of the classical $SL(2, \mathbb{R})$ gauge transformations following an analysis similar to the one made in [8], motivated by several reasons. Firstly, the analysis in [8] is not completely explicit, in fact authors focused mainly on the $ISO(2)$ case. Secondly, the analysis makes use of the three principal moments of inertia I_i ($i = 1, 2, 3$) as independent parameters, however it was recognized long ago, that separability of the Laplace operator for the general asymmetric Eulerian top can be achieved only if the analysis is made in spheroconal coordinates where one introduces related dimensionless inertia parameters e_i ($i = 1, 2, 3$) [11–14]. Thirdly, in analogy to the Nambu's result about the $SL(3, \mathbb{R})$ canonical transformations where he found that the inequivalent Lie-Poisson structures are the $SO(4)$ Lie algebra and all the ones related to it via analytical continuation or group contraction, it is natural to expect that the $SL(2, \mathbb{R})$ gauge symmetry allows as a Lie-Poisson structure all the algebras related to $SO(3)$ via analytical continuation and group contractions. Indeed this is the conclusion in [10] whereas the case $ISO(1, 1)$ is not reported in [8]. As a result of our analysis we have identified this algebra as a possible Lie-Poisson structure for an asymmetric extended rigid body and we show that this algebra is not possible for a symmetric body. As a by-product, we include the explicit form of the generators for the different possibilities of Lie algebras, this is something that for the best of our knowledge is missing in the literature, and in this contribution we fill in this gap. We include in our discussion the Bianchi nomenclature for the different Lie algebras [15]. Finally it would be very interesting to find the relation between the Nambu dynamics and some recent results in the area of atomic physics and asymmetric rotational molecular spectrums [16–19]. This issue will be addressed elsewhere.

Relevant for our discussion is the geometrical method introduced by Poincaré for visualizing the motion of the endpoint of the angular momentum vector \vec{L} , which for the Eulerian top takes place in the intersection of the ellipsoid of energy E and the sphere of angular momentum G . This construction is automatically incorporated in the Nambu formulation, because $\dot{E} = \dot{G} = 0$, and therefore both E and G are constants of the motion implying that the orbit of the system in phase space is determined as the intersection of the two surfaces $E = \text{const.}$ and $G = \text{const.}$ Analytically the parameterization of these intersections is given in terms of Jacobi elliptic functions whose modulus depend on the values of the three inertia parameter e_i ($i = 1, 2, 3$) only (see for instance [14] and references therein). In fact, working in these coordinates is the only known way that allows to quantize the asymmetric free rigid body [11, 12, 20–22]. In these coordinates the geometry of the energy ellipsoid E is replaced by a Casimir function h with the geometry of an elliptic hyperboloid that can be either of one or two sheets depending of the numerical value of the quotient $2E/G$ or it can also have the geometry of an elliptic cone or a hyperbolic cylinder in the proper limit situations. This geometrical richness of the surfaces h gives more geometrical possibilities when the Poincaré construction is performed. In this regard our analysis is close to the one performed for the Maxwell-Bloch system [23].

Our exposition is self-contained as possible. In section II we introduce the rigid body system and in section III we discuss the issue of the Casimir functions. Section IV exposes the classification of the different geometries of the Casimir functions and section V deals with the solutions of the Euler equations. In section VI we discuss the $SL(2, \mathbb{R})$ gauge symmetry of the Eulerian top and in VII we give in detail all the different inequivalent Lie algebras for which the Euler equations are the equations of motion. We give our conclusions in VIII.

II. THE RIGID BODY

In a body-fix reference frame the torque free rigid body motion is governed by the Euler equations

$$\frac{d\vec{L}}{dt} = \vec{L} \times I^{-1}\vec{L}, \quad (1)$$

where \vec{L} is the vector of angular momentum and I the moment of inertia tensor. If the body-fixed frame is oriented to coincide with the principal axes of inertia, the matrix I is diagonal. Without losing generality in the following discussion we consider that the principal moments of inertia satisfy the inequality

$$I_1 < I_2 < I_3. \quad (2)$$

When written in a basis, the L_i components of the angular momentum are the generators of a $SO(3)$ Lie algebra

$$[L_i, L_j] = \varepsilon_{ijk} L_k. \quad (3)$$

The problem has two integrals of motion, the kinetic energy E and the square of the angular momentum L^2 , which are given in terms of the moments of inertia and the components of \vec{L} as

$$E = \frac{L_1^2}{2I_1} + \frac{L_2^2}{2I_2} + \frac{L_3^2}{2I_3}, \quad (4)$$

and

$$L^2 = L_1^2 + L_2^2 + L_3^2, \quad (5)$$

respectively. In a Euclidean space whose coordinates are the components of the angular momentum, the equation for the energy represents an ellipsoid with semiaxes $\sqrt{2EI_1}$, $\sqrt{2EI_2}$ and $\sqrt{2EI_3}$, whereas the equation for the conservation of the modulus of the angular momentum represents a sphere of radius L . Because $2EI_1 < L^2 < 2EI_3$, the radius of the sphere has a value between the minimum and maximum values of the semiaxes of the ellipsoid. The dynamical problem is usually solved using the Poincot construction. When the vector \vec{L} moves relative to the axes of inertia of the top, it lies along the curve of intersection of the surfaces $E = \text{constant}$ and $L^2 = \text{constant}$. It is clear that the solutions of the Euler equations (1) will depend of five different parameters, the three moments of inertia, the energy and the square of the angular momentum. However it has been shown that the problem can be rewritten in such a way that the solutions will depend only on two parameters. We will return to this point afterwards.

III. CASIMIR FUNCTIONS

Because the Nambu dynamics in three dimensions requires of two ‘‘Hamiltonians’’ or Casimir functions, we start our discussion reminding this issue in our problem at hand. For the asymmetric rigid body the three principal moments of inertia are all different and the proper coordinates system that describes it are the spheroconal ones [11]. These are the coordinates implied naturally by the two quadratic Casimir functions of the system: L^2 and $e_0 \equiv e_1 L_1^2 + e_2 L_2^2 + e_3 L_3^2$, where the dimensionless coefficients e_i ’s ($i = 1, 2, 3$) are restricted by two conditions as we will discuss below. In the degenerated cases where two principal moments of inertia are equal (symmetric case) or the three are equal (spherical case), the spherical coordinates are the ones that naturally incorporate the canonical Casimir functions of the system: L^2 and L_3 . Notice that in the asymmetric case the two Casimir functions are quadratic in L_i^2 , whereas in the symmetric case one of them is linear. This issue was systematically addressed in [24] and because our analysis strongly depends on the geometry of the Casimir functions, here we summarize the main steps to determine these functions.

The starting point is to recognize that any first order vector in the $O(3)$ algebra can be rotated into say L_3 , and therefore the basis: L^2, L_3 , will constitute a canonical basis. However the asymmetric case is not properly described by this basis and in order to find the proper basis, it is necessary to consider symmetric second order tensors of the form

$$N = \sum_{ij} N_{ij} L_i L_j, \quad (6)$$

in the enveloping algebra $O(3)$, an example of such a tensor is the kinetic energy E (4). Here N is a real symmetric matrix and working in the representation where it is diagonal, their eigenvalues are given by the inverse of the principal moments of inertia $N_{ii} \equiv 1/I_i$. The classification depends on the number of different eigenvalues which leads to three different cases, two of which are described by the canonical basis, and the remaining one by a basis of two quadratic functions. The cases are the following:

- *Spherical case:* If $I_1 = I_2 = I_3$, the tensor $N = L^2/I_1$ is written in terms of the canonical basis.
- *Symmetric case:* Without losing generality, if $I_1 = I_2 \neq I_3$, the tensor $N = L^2/I_1 + L_3^2/(I_3 - I_1)$ is written in terms of the canonical basis.
- *Asymmetric case:* If $I_1 \neq I_2 \neq I_3$ the tensor $N = L^2/I_1 + (1/I_1 - 1/I_3)(k_1^2 L_1^2 + k_2^2 L_2^2 + L_3^2)$, and the basis is different from the canonical. In fact in addition to L^2 the second quadratic tensors can be taken either as: $k_2^2 L_2^2 + L_3^2$, $L_1^2 + k_1^2 L_2^2$ or $k_2^2 L_1^2 - k_1^2 L_3^2$. Here

$$k_1^2 \equiv \frac{\frac{1}{I_2} - \frac{1}{I_3}}{\frac{1}{I_1} - \frac{1}{I_3}} \quad \text{and} \quad k_2^2 \equiv \frac{\frac{1}{I_1} - \frac{1}{I_2}}{\frac{1}{I_1} - \frac{1}{I_3}}, \quad (7)$$

which takes values in the rank $0 < k_1^2 < 1$, $0 < k_2^2 < 1$ and satisfy $k_1^2 + k_2^2 = 1$.

In the asymmetric case it is clear that only one quadratic function besides L^2 is needed to complete the basis because for instance $L_3^2 + k_2^2 L_2^2 = L^2 - (L_1^2 + k_1^2 L_2^2)$. We emphasize that the second tensor depends of one real parameter

only, for instance k_1^2 . Following [12, 13] we chose to rewrite the second quadratic tensor by expressing the matrix N in terms of its irreducible representations: its trace and a traceless symmetric matrix, obtaining

$$N = \left(N_{11} - \frac{1}{3}\text{Tr } N\right) L_1^2 + \left(N_{22} - \frac{1}{3}\text{Tr } N\right) L_2^2 + \left(N_{33} - \frac{1}{3}\text{Tr } N\right) L_3^2 + \frac{1}{3}\text{Tr } N L^2. \quad (8)$$

It is possible to replace the set of 3 principal moments of inertia $I_i = 1/N_{ii}$ by 3 equivalent *inertia parameters* $(\bar{e}_1, \bar{e}_2, \bar{e}_3)$, which have dimension of the inverse of a moment of inertia. The main difference between these two sets is that the \bar{e}_i 's can be positive, negative or even zero, whereas the I_i 's are always positive. Writing the diagonal components of the traceless symmetric part of N as [14]

$$N_{11} - \frac{1}{3}\text{Tr } N \equiv \bar{e}_1, \quad N_{22} - \frac{1}{3}\text{Tr } N \equiv \bar{e}_2, \quad N_{33} - \frac{1}{3}\text{Tr } N \equiv \bar{e}_3, \quad (9)$$

the traceless condition provides a constraint for the 3 parameters \bar{e}_i 's

$$\bar{e}_1 + \bar{e}_2 + \bar{e}_3 = 0. \quad (10)$$

The second constraint on the inertia parameters is obtained by computing the square of the traceless matrix and then computing its trace, which leads to the condition

$$\frac{1}{3} \left[(N_{11} - N_{22})^2 + (N_{22} - N_{33})^2 + (N_{33} - N_{11})^2 \right] = (\bar{e}_1^2 + \bar{e}_2^2 + \bar{e}_3^2). \quad (11)$$

These two conditions on the inertia parameters are respectively the expressions for both the first and second order symmetric invariant functions of the roots of a third order polynomial equation. The invariance is respect to the Galois group which for a third order polynomial is the symmetric group S_3 or the permutation group P_3 . Using the standard notation of the Weierstrass elliptic functions, a third order polynomial equation can always be cast after a Tschirnhaus transformation to the form $4y^3 - g_2y - g_3 = 4(y - \bar{e}_1)(y - \bar{e}_2)(y - \bar{e}_3) = 0$, in fact the transformation (9) is a Tschirnhaus transformation applied to the cubic characteristic polynomial associated to the matrix N_{ij} of the equation (6) (see for instance [25]). The three symmetric invariant functions of the roots are equation (10), $g_2 = -4(\bar{e}_1\bar{e}_2 + \bar{e}_2\bar{e}_3 + \bar{e}_3\bar{e}_1) = 2(\bar{e}_1^2 + \bar{e}_2^2 + \bar{e}_3^2)$ and $g_3 = 4\bar{e}_1\bar{e}_2\bar{e}_3$. The cubic polynomial has three real roots only if $g_2 > 0$ which is clearly satisfied by the change (9) because from (11)

$$g_2 \equiv \frac{2}{3} \left[\left(\frac{1}{I_1} - \frac{1}{I_2} \right)^2 + \left(\frac{1}{I_2} - \frac{1}{I_3} \right)^2 + \left(\frac{1}{I_3} - \frac{1}{I_1} \right)^2 \right] > 0. \quad (12)$$

The third order symmetric invariant function g_3 can have either sign or it can be zero. It is well known that the roots \bar{e}_i 's can be written in terms of the invariants of the theory. Because the invariant of first order is zero, the solutions are given in terms of the invariants g_2 and g_3 in the form (see for instance [26])

$$\bar{e}_1 = -\sqrt{\frac{g_2}{3}} \cos\left(\frac{\varphi}{3}\right), \quad \bar{e}_2 = \sqrt{\frac{g_2}{3}} \cos\left(\frac{\varphi + \pi}{3}\right), \quad \bar{e}_3 = \sqrt{\frac{g_2}{3}} \cos\left(\frac{\varphi - \pi}{3}\right), \quad (13)$$

where the angular variable $\varphi \equiv \arccos(-(3/g_2)^{3/2}g_3)$. This form of the solutions is related to the ones in [14] by introducing the *asymmetry parameter* $\kappa = \pi + \varphi/3$. Even more, we can introduce a related set of *dimensionless inertia parameters* e_i 's through the relation $\bar{e}_i \equiv \sqrt{g_2/3} e_i$, namely

$$e_1 = \cos(\kappa), \quad e_2 = \cos(\kappa - 2\pi/3), \quad e_3 = \cos(\kappa + 2\pi/3), \quad (14)$$

which satisfy the conditions

$$e_1 + e_2 + e_3 = 0, \quad e_1^2 + e_2^2 + e_3^2 = 3/2 \quad \text{and} \quad e_1 e_2 e_3 = (\cos 3\kappa)/4. \quad (15)$$

In this way we obtain finally that any second order tensor N can be expressed as a combination of the two quadratic Casimir functions L^2 and $e_1 L_1^2 + e_2 L_2^2 + e_3 L_3^2$ in the way

$$N = \frac{1}{3}\text{Tr } N L^2 + \sqrt{\frac{g_2}{3}} (e_1 L_1^2 + e_2 L_2^2 + e_3 L_3^2). \quad (16)$$

From the geometrical point of view, in the three dimensional space of dimensionless inertia parameters e_i 's, for the general asymmetric case the first condition in (15) defines a plane that cross the origin, whereas the second

one represents a sphere of square radius equal to $3/2$. The intersection of these two surfaces is a circle which is parameterized by the angular asymmetry parameter κ . The role of the invariant g_2 is to change the size of the sphere and therefore of the intersecting circle whereas the role of the invariant g_3 is to parameterize the angular variable κ that describes the intersecting circle. It is clear that given a set of values for the three principal moments of inertia I_i , the values of $\text{Tr}N$, g_2 and g_3 , and therefore of the angular variable κ are completely determined. Regarding the asymmetry parameter notice that when it takes values in the interval $\kappa \in [0, \pi/3]$, the inertia parameters have the order

$$e_3 \leq e_2 \leq e_1, \quad (17)$$

with $e_3 \neq e_1$ (equivalent to condition (2)). In general as expected, for $\kappa \in [0, 2\pi]$, we obtain all the six possible orders for the three inertia parameters e_i 's, due to the fact that the inertia parameters are given in terms of invariants under the Galois symmetric group S_3 (see Figure 1) and the group is of order six. In particular, when $\kappa = n\pi/3$, $n = 1, \dots, 6$, the rigid body is symmetric (two inertia parameters equal), and when $\kappa = (2n-1)\pi/6$, $n = 1, \dots, 6$ the rigid body is the most asymmetric one (one inertia parameter null). For any value of κ , at least one of the inertia parameters is positive, another is negative, and the third one may be either positive, null or negative.

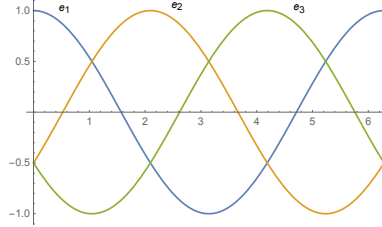


FIG. 1: Dimensionless inertia parameters e_1 , e_2 and e_3 for different values of the asymmetry parameter $\kappa \in [0, 2\pi]$.

In terms of the e_i 's the parameters k_1^2 and k_2^2 defined in (7) are rewritten respectively as

$$k_1^2 = \frac{e_2 - e_3}{e_1 - e_3}, \quad \text{and} \quad k_2^2 = \frac{e_1 - e_2}{e_1 - e_3}. \quad (18)$$

A consequence of a change in the order of the inertia parameters is a change in the expression for these quotients. For instance, if the asymmetry parameter takes values in the interval $\kappa \in (\pi/3, 2\pi/3)$ the new parameters ordering is $e_3 < e_1 < e_2$ and the parameters analogous to (18) are obtained from them by simply interchange $e_1 \leftrightarrow e_2$, obtaining $(e_1 - e_3)/(e_2 - e_3) = 1/k_1^2$ and $(e_2 - e_1)/(e_2 - e_3) = -k_2^2/k_1^2$. In the following table we present the six cases.

Asymmetry parameter interval	Order of the inertia parameters		
$\kappa \in (0, \pi/3)$	$e_3 < e_2 < e_1$	k_1^2	k_2^2
$\kappa \in (\pi/3, 2\pi/3)$	$e_3 < e_1 < e_2$	$\frac{1}{k_1^2}$	$-\frac{k_2^2}{k_1^2}$
$\kappa \in (2\pi/3, \pi)$	$e_1 < e_3 < e_2$	$\frac{1}{k_2^2}$	$-\frac{k_1^2}{k_2^2}$
$\kappa \in (\pi, 4\pi/3)$	$e_1 < e_2 < e_3$	k_2^2	k_1^2
$\kappa \in (4\pi/3, 5\pi/3)$	$e_2 < e_1 < e_3$	$-\frac{k_2^2}{k_1^2}$	$\frac{1}{k_1^2}$
$\kappa \in (5\pi/3, 2\pi)$	$e_2 < e_3 < e_1$	$-\frac{k_1^2}{k_2^2}$	$\frac{1}{k_2^2}$

TABLE I: List of the six different ordering of the inertia parameters for different intervals of the asymmetry parameter κ . We include the change in the quotients (18). Notice that in every case the analogous restriction to $k_1^2 + k_2^2 = 1$ takes place.

IV. THE GEOMETRY OF THE CASIMIR FUNCTIONS

Notice that the quotient E/L^2 has dimension of the inverse of moment of inertia, therefore we can define the dimensionless parameter e_0 associated with the energy and the square of the angular momentum in a similar form to

(9) as

$$\frac{2E}{L^2} = \frac{1}{3}\text{Tr}N + \sqrt{\frac{g_2}{3}}e_0. \quad (19)$$

We call to e_0 the energy parameter for short. Introducing the unitary vector \vec{u} in the direction of the angular momentum \vec{L} : $\vec{u} \equiv \vec{L}/L$ we can rewrite the angular momentum sphere (5) as

$$u_1^2 + u_2^2 + u_3^2 = 1. \quad (20)$$

Taking $N = 2E$ in equation (16) and combining it with (19), we obtain the dimensionless form of the another second order Casimir function

$$e_1 u_1^2 + e_2 u_2^2 + e_3 u_3^2 = e_0. \quad (21)$$

In the dimensionless angular momentum space (u_1, u_2, u_3) , the surface (20) is a unitary sphere, whereas the geometry of the Casimir function (21) depends on the number of positive and negative coefficients in the equation. In the following we give a classification of the different possible geometries for this surface. We emphasize that this surface does not correspond to the energy surface (4) but is related to it through the transformation (19). Nevertheless abusing of the language we call to e_0 the dimensionless energy parameter and to the Casimir function (21) the energy surface. A close classification to the one we present here was discussed in the context of quantization using the principle of correspondence in [21].

1. Asymmetric rigid bodies

In the following without losing generality we will assume that the inertia parameters satisfy inequalities (17). The condition for intersection between the sphere of angular momentum and the energy surface becomes in these parameters

$$e_3 < e_0 < e_1. \quad (22)$$

The geometry of the energy surface depends on the value of the energy parameter e_0 respect to the inertia parameter e_2 . There are two different cases and they correspond to:

- Case $e_2 < e_0 < e_1$

Generically the surface (21) is a hyperboloid instead of the original ellipsoid of energy (4), because at least one inertia parameter is positive (e_1 for the ordering (2)) and another is negative (e_3). As can be seen in figure 1, the parameter e_2 can be either, positive, zero or negative. Therefore because $e_2 < e_0$ the energy parameter can also be either positive, zero or negative. In table II we give all different possible geometries of the Casimir function (21).

e_2	e_0	“Energy” surface equation	“Energy” surface geometry
< 0	< 0	$-\frac{e_1}{ e_0 }u_1^2 + \frac{ e_2 }{ e_0 }u_2^2 + \frac{ e_3 }{ e_0 }u_3^2 = 1$	elliptic hyperboloid of one sheet around u_1
< 0	$= 0$	$e_1 u_1^2 = e_2 u_2^2 + e_3 u_3^2$	elliptic cone around u_1
< 0	> 0	$\frac{e_1}{e_0}u_1^2 - \frac{ e_2 }{e_0}u_2^2 - \frac{ e_3 }{e_0}u_3^2 = 1$	elliptic hyperboloid of two sheets around u_1
$= 0$	> 0	$\frac{e_1}{e_0}(u_1^2 - u_3^2) = 1$	hyperbolic cylinder
> 0	> 0	$\frac{e_1}{e_0}u_1^2 + \frac{e_2}{e_0}u_2^2 - \frac{ e_3 }{e_0}u_3^2 = 1$	elliptic hyperboloid of one sheet around u_3

TABLE II: List of the different geometries the Casimir function (21) can describe, in the case where the dimensionless energy parameter e_0 take values in the interval $e_2 < e_0 < e_1$.

- Case $e_3 < e_0 < e_2$

In this case the energy surfaces are very similar to the previous case. The main difference is the orientation of the surfaces. In table III we list the different geometries of the energy surface.

e_0	e_2	“Energy” surface equation	“Energy” surface geometry
< 0	< 0	$-\frac{e_1}{ e_0 }u_1^2 + \frac{ e_2 }{ e_0 }u_2^2 + \frac{ e_3 }{ e_0 }u_3^2 = 1$	elliptic hyperboloid of one sheet around u_1
< 0	$= 0$	$-\frac{e_1}{ e_0 }(u_1^2 - u_3^2) = 1$	hyperbolic cylinder
< 0	> 0	$-\frac{e_1}{ e_0 }u_1^2 - \frac{e_2}{ e_0 }u_2^2 + \frac{ e_3 }{ e_0 }u_3^2 = 1$	elliptic hyperboloid of two sheets around u_3
$= 0$	> 0	$e_1u_1^2 + e_2u_2^2 = e_3 u_3^2$	elliptic cone around u_3
> 0	> 0	$\frac{e_1}{e_0}u_1^2 + \frac{e_2}{e_0}u_2^2 - \frac{ e_3 }{e_0}u_3^2 = 1$	elliptic hyperboloid of one sheet around u_3

TABLE III: List of the different geometries the Casimir function (21) can describe, in the case where the dimensionless energy parameter e_0 take values in the interval $e_3 < e_0 < e_2$.

2. Symmetric rigid bodies

Important throughout our exposition are the special cases when the rigid body has cylindric symmetry. These cases can be obtained as limiting situations of the general asymmetric rigid body taking two inertia parameters equal. There are two of such limits in the parameters ordering (17) we are discussing here. The so called *prolate* limit $e_3 = e_2$ and the *oblate* limit $e_2 = e_1$.

- Prolate case, $e_3 = e_2 < 0 < e_1$ ($\kappa = 0$): The smallest principal momentum of inertia is I_1 . We can take the prolate limit in 3 different situations. The geometries of the Casimir function (21) are given in table IV.

e_0	“Energy” surface equation	“Energy” surface geometry
< 0	$-\frac{e_1}{ e_0 }u_1^2 + \frac{ e_3 }{ e_0 }(u_2^2 + u_3^2) = 1$	hyperboloid of revolution of one sheet around u_1
$= 0$	$e_1u_1^2 = e_3 (u_2^2 + u_3^2)$	circular cone around u_1
> 0	$\frac{e_1}{e_0}u_1^2 - \frac{ e_3 }{e_0}(u_2^2 + u_3^2) = 1$	hyperboloid of revolution of two sheets around u_1

TABLE IV: List of the different geometries that the Casimir surface (21) can take in the symmetric prolate case.

- Oblate case, $e_3 < 0 < e_2 = e_1$ ($\kappa = \pi/3$): The largest principal momentum of inertia momentum I_3 . The oblate limit also occurs in 3 different situations. The geometries of the Casimir function (21) are given in the table V.

e_0	“Energy” surface equation	“Energy” surface geometry
< 0	$-\frac{e_1}{ e_0 }(u_1^2 + u_2^2) + \frac{ e_3 }{ e_0 }u_3^2 = 1$	hyperboloid of revolution of two sheets around u_3
$= 0$	$e_1(u_1^2 + u_2^2) = e_3 u_3^2$	circular cone around u_3
> 0	$\frac{e_1}{e_0}(u_1^2 + u_2^2) - \frac{ e_3 }{e_0}u_3^2 = 1$	hyperboloid of revolution of one sheet around u_3

TABLE V: List of the different geometries that the Casimir surface (21) can take in the symmetric oblate case.

V. DIMENSIONLESS EULER EQUATIONS AND ITS SOLUTIONS

It is straightforward to write down the Euler equations (1) in terms of both the dimensionless inertia parameters e_i 's and the dimensionless vector of angular momentum \vec{u} . In components form we have

$$\frac{du_i}{dt} = \frac{1}{2}\sqrt{\frac{g_2}{3}}L\varepsilon_{ijk}(e_k - e_j)u_ju_k. \quad (23)$$

It is possible to absorb the factor $\sqrt{\frac{g_2}{3}}L$, by defining a dimensionless time parameter: $x \equiv t\sqrt{\frac{g_2}{3}}L$, obtaining finally

$$\dot{u}_i = \frac{1}{2}\varepsilon_{ijk}(e_k - e_j)u_ju_k, \quad (24)$$

where $(\dot{\cdot})$ denotes derivative with respect to the dimensionless time x . The solutions to these equations in the general asymmetric case are given in terms of elliptic functions. From the classical mechanics point of view there are at least two different but equivalent ways to solve them. One method of solution consists in applying the Poincaré construction, *i.e.* the solutions are obtained as the intersection of the energy and angular momentum surfaces. In [27] the solutions are obtained for the Casimir functions (4)-(5) whereas in [13] the solutions are obtained for the Casimir functions

(20)-(21). The second form of solving the equations is known as the separation method (see for instance [28, 29] and references therein) which is the relevant method when one deals with the quantization of the asymmetric rigid body. Here we are not going into details, but for purposes of completeness in our exposition here we give the explicit solution obtained by both methods, details can be found for instance in [13, 28].

I. Case $e_3 < e_2 < e_0 < e_1$ (i.e. $1/I_2 < 2E/L^2$).

When the energy parameter e_0 is between the inertia parameters e_2 and e_1 , the solutions are given by

$$u_1(\tau) = \text{sn}(\tau', m_c) \text{dn}(\tau, m), \quad u_2(\tau) = \text{dn}(\tau', m_c) \text{sn}(\tau, m), \quad u_3(\tau) = \text{cn}(\tau', m_c) \text{cn}(\tau, m), \quad (25)$$

where τ is a dimensionless time parameter defined as

$$\tau = x \sqrt{(e_1 - e_2)(e_0 - e_3)}. \quad (26)$$

Here the amplitudes of the solutions are written as Jacobi elliptic functions at parameter $\tau' = \text{constant}$ and are related to the dimensionless parameters e_i and e_0 in the form

$$\text{sn}^2(\tau', m_c) = \frac{e_0 - e_3}{e_1 - e_3}, \quad \text{cn}^2(\tau', m_c) = \frac{e_1 - e_0}{e_1 - e_3}, \quad \text{dn}^2(\tau', m_c) = \frac{e_1 - e_0}{e_1 - e_2}. \quad (27)$$

The square modulus of the elliptic functions is

$$m^2 \equiv \frac{(e_2 - e_3)(e_1 - e_0)}{(e_0 - e_3)(e_1 - e_2)} = \frac{(e_1 - e_0)}{(e_0 - e_3)} \frac{k_1^2}{k_2^2}. \quad (28)$$

which takes values in the interval $0 < m^2 < 1$. m_c is its complementary modulus $m_c^2 = 1 - m^2$.

II. Case $e_3 < e_0 < e_2 < e_1$ (i.e. $2E/L^2 < 1/I_2$).

In this case the solutions are given by

$$u_1 = \text{cn}\left(\tau', i \frac{m_c}{m}\right) \text{cn}\left(m\tau, \frac{1}{m}\right), \quad u_2 = \text{dn}\left(\tau', i \frac{m_c}{m}\right) \text{sn}\left(m\tau, \frac{1}{m}\right), \quad u_3 = \text{sn}\left(\tau', i \frac{m_c}{m}\right) \text{dn}\left(m\tau, \frac{1}{m}\right). \quad (29)$$

where the square modulus of the elliptic function take values in the interval $0 < 1/m^2 < 1$. Here m^2 was defined in (28) but due to the fact that $e_3 < e_0 < e_2 < e_1$, $m^2 > 1$. The time parameter τ is the one defined in (26).

Physical interpretation of the solutions is straightforward, the curves (25)-(29) are the parameterization of the intersection of the unitary sphere of angular momentum and the corresponding surface of the Casimir function (21), which for the asymmetric case is given in tables II and III. It is important to accentuate that both the dimensionless time parameter τ and the modulus m in the elliptic functions depend explicitly on the inertia parameters e_i 's and the energy parameter e_0 . This fact implies for instance that the parameterization of the curves on the angular momentum sphere (20) depends on the value of e_0 . However, in principle it should be possible to parameterize these curves in terms of elliptic functions whose modulus does not depend of this parameter, because the sphere represents the surface of angular momentum only. This parameterization exists and it is mandatory to use it in the quantization of the asymmetric top because in this basis, it is possible to achieve separability of the Laplacian operator in the Liouville sense (see for instance [28] for a discussion of this property). For this reason the solution method is called the separation method.

The parameterization of the solutions is obtained by using a first degree transformation of the Jacobi elliptic functions [30]. A detailed derivation of the change of basis is discussed in [28, 29], and is given explicitly by

$$\text{sn}(\tau_1, m) = \frac{k_2 \text{sn}(\chi_1, k_1)}{\text{dn}(\chi_1, k_1)}, \quad \text{and} \quad m \text{sn}(\tau_2, m) = -\frac{k_1 \text{sn}(\chi_2, k_2)}{\text{dn}(\chi_2, k_2)}. \quad (30)$$

There are similar transformations for the other two Jacobi functions cn and dn . In the new parameterization the time parameter τ (26), is written as a difference of two new time parameters τ_1 and τ_2 , i.e. $\tau = \tau_1 - \tau_2$ and the new angular parameters χ_1 and χ_2 depend on the dimensionless inertia parameters as

$$\chi_1 \equiv \sqrt{e_1 - e_3} \phi_1, \quad \text{and} \quad \chi_2 \equiv \sqrt{e_1 - e_3} \phi_2. \quad (31)$$

The modulus k_1 and k_2 of the elliptic functions are given by equations (18). Substitution of transformations (30) plus the ones for the other Jacobi functions into equations (25) leads to a parameterization of the unitary S^2 sphere in terms of spheroconal coordinates that does not depend explicitly on e_0

$$u_1 = \text{dn}(\chi_1, k_1) \text{sn}(\chi_2, k_2), \quad u_2 = \text{cn}(\chi_1, k_1) \text{cn}(\chi_2, k_2), \quad u_3 = \text{sn}(\chi_1, k_1) \text{dn}(\chi_2, k_2), \quad (32)$$

where the angular parameters are defined in the intervals

$$-2K_1 \leq \chi_1 \leq 2K_1, \quad \text{and} \quad -K_2 \leq \chi_2 \leq K_2, \quad (33)$$

with K_1 and K_2 the corresponding quarter periods of the Jacobi elliptic functions [24]. Because

$$\frac{u_1^2}{\text{dn}^2(\chi_1, k_1)} + \frac{u_2^2}{\text{cn}^2(\chi_1, k_1)} = 1, \quad \text{and} \quad \frac{u_2^2}{\text{cn}^2(\chi_2, k_2)} + \frac{u_3^2}{\text{dn}^2(\chi_2, k_2)} = 1, \quad (34)$$

the coordinates χ_1 and χ_2 are elliptical coordinates on the sphere with axis along the u_1 and u_3 direction respectively. From here onwards we will work in the parameterization (32).

VI. $SL(2, \mathbb{R})$ SYMMETRIES OF THE EULER EQUATIONS

In order to make manifest the gauge symmetries of the Euler equations, we notice that its dimensionless form

$$\dot{\vec{u}} = \vec{u} \times \epsilon \vec{u} \quad (35)$$

with ϵ a diagonal matrix of the form $\epsilon = \text{diag}(e_1, e_2, e_3)$, can be rewritten as the gradient of two scalar functions

$$\dot{\vec{u}} = \nabla l \times \nabla h, \quad (36)$$

where

$$h = \frac{1}{2}(e_1 u_1^2 + e_2 u_2^2 + e_3 u_3^2) - \frac{1}{2}e_0 = 0 \Rightarrow \epsilon \vec{u} = \nabla h = (e_1 u_1, e_2 u_2, e_3 u_3), \quad (37)$$

$$l = \frac{1}{2}(u_1^2 + u_2^2 + u_3^2) - \frac{1}{2} = 0 \Rightarrow \vec{u} = \nabla l = (u_1, u_2, u_3). \quad (38)$$

This form of writing the Euler equations makes explicit its invariance under any $SL(2, \mathbb{R})$ transformation [8]. It is straightforward to check that the transformation

$$\begin{pmatrix} H \\ N \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} h \\ l \end{pmatrix}, \quad (39)$$

with $ad - bc = 1$ leads to

$$\dot{\vec{u}} = \nabla N \times \nabla H. \quad (40)$$

The Nambu dynamics established that, given a dynamical system with Hamiltonian H , any dynamical quantity Q evolves with time according to

$$\dot{Q} = \{Q, N, H\} = X_H Q, \quad (41)$$

where the generator X_G is given by

$$X_G = (\nabla N \times \nabla G) \cdot \nabla. \quad (42)$$

In the spheroconal coordinates (32), the gradient operator has components

$$\begin{aligned} \frac{\partial}{\partial u_1} &= \frac{1}{\tilde{h}^2} \left[\text{dn}(\chi_1, k_1) \text{cn}(\chi_2, k_2) \text{dn}(\chi_2, k_2) \frac{\partial}{\partial \chi_1} + k_1^2 \text{sn}(\chi_1, k_1) \text{cn}(\chi_1, k_1) \text{sn}(\chi_2, k_2) \frac{\partial}{\partial \chi_2} \right], \\ \frac{\partial}{\partial u_2} &= \frac{1}{\tilde{h}^2} \left[-\text{cn}(\chi_1, k_1) \text{sn}(\chi_2, k_2) \text{dn}(\chi_2, k_2) \frac{\partial}{\partial \chi_1} + \text{sn}(\chi_1, k_1) \text{dn}(\chi_1, k_1) \text{cn}(\chi_2, k_2) \frac{\partial}{\partial \chi_2} \right], \\ \frac{\partial}{\partial u_3} &= \frac{1}{\tilde{h}^2} \left[-k_2^2 \text{sn}(\chi_1, k_1) \text{sn}(\chi_2, k_2) \text{cn}(\chi_2, k_2) \frac{\partial}{\partial \chi_1} + \text{cn}(\chi_1, k_1) \text{dn}(\chi_1, k_1) \text{dn}(\chi_2, k_2) \frac{\partial}{\partial \chi_2} \right], \end{aligned} \quad (43)$$

where the function \tilde{h} is defined as

$$\tilde{h}^2 \equiv \tilde{h}^2(\chi_1, \chi_2) = 1 - k_1^2 \text{sn}^2(\chi_1, k_1) - k_2^2 \text{sn}^2(\chi_2, k_2). \quad (44)$$

According with the transformation (39), the generator X_G in components form is expressed as

$$X_G = (ce_1 + d)u_1 \left(\frac{\partial G}{\partial u_2} \frac{\partial}{\partial u_3} - \frac{\partial G}{\partial u_3} \frac{\partial}{\partial u_2} \right) + (ce_2 + d)u_2 \left(\frac{\partial G}{\partial u_3} \frac{\partial}{\partial u_1} - \frac{\partial G}{\partial u_1} \frac{\partial}{\partial u_3} \right) + (ce_3 + d)u_3 \left(\frac{\partial G}{\partial u_1} \frac{\partial}{\partial u_2} - \frac{\partial G}{\partial u_2} \frac{\partial}{\partial u_1} \right). \quad (45)$$

In order to determine the structure of the Lie algebra, we calculate the Lie-Poisson bracket for the Hamiltonian vector fields associated with the coordinate functions u_i

$$\begin{aligned} X_{u_1} &= (ce_3 + d)u_3 \partial_2 - (ce_2 + d)u_2 \partial_3, \\ X_{u_2} &= (ce_1 + d)u_1 \partial_3 - (ce_3 + d)u_3 \partial_1, \\ X_{u_3} &= (ce_2 + d)u_2 \partial_1 - (ce_1 + d)u_1 \partial_2. \end{aligned} \quad (46)$$

Explicitly the Lie-Poisson brackets are given by

$$[X_{u_1}, X_{u_2}] = (ce_3 + d)X_{u_3}, \quad [X_{u_2}, X_{u_3}] = (ce_1 + d)X_{u_1}, \quad [X_{u_3}, X_{u_1}] = (ce_2 + d)X_{u_2}. \quad (47)$$

The classification of the different Lie algebras arise from the $SL(2, \mathbb{R})$ invariant orbits. Before we address this point, it is convenient to remind that for the symmetric cases the suitable coordinates are the spherical coordinates. We can obtain the corresponding generators from (32) in the proper limits. For the oblate case ($e_1 = e_2$) we have, due to the fact that Jacobi elliptic functions go to hyperbolic (circular) functions for a value of the modulus $k_1 = 1$ ($k_2 = 0$), that the coordinates (32) become

$$u_1 = \operatorname{sech}(\chi_1) \sin(\chi_2), \quad u_2 = \operatorname{sech}(\chi_1) \cos(\chi_2), \quad u_3 = \tanh(\chi_1).$$

Geometrically in this limit the ellipses $\chi_1 = \text{constant}$ become circles and therefore the coordinate χ_2 can be renamed as the standard azimuthal angle $\chi_2 \equiv \phi$. The half piece of the ellipses $\chi_2 = \text{constant}$ degenerate into the semi-circle intersection of the half plane: $u_2 = 0$, $u_1 > 0$, with the sphere. Because the hyperbolic function $\operatorname{sech}(\chi_1)$ is always positive unlike the elliptic Jacobi function $\operatorname{cn}(\chi_1)$, in order to cover the whole sphere, the angle ϕ must be extended to take values in the interval $[-\pi, \pi]$ instead of the interval (33) for χ_1 . It is common to parametrize the unitary S^2 sphere with circular functions instead of hyperbolic ones, thus replacing $\operatorname{sech}(\chi_1) \rightarrow \sin(\theta) \Rightarrow \tanh(\chi_1) \rightarrow \cos(\theta)$ with $\theta \in [0, \pi]$, we obtain

$$u_1 = \sin \theta \sin \phi, \quad u_2 = \sin \theta \cos \phi, \quad u_3 = \cos \theta. \quad (48)$$

As a consequence we have from (43) that the components of the gradient operator in the oblate case are

$$\begin{aligned} \partial_1 &= \sin \phi \cot \theta \frac{\partial}{\partial \phi} + \cos \phi \csc \theta \frac{\partial}{\partial \theta}, \\ \partial_2 &= \cos \phi \cot \theta \frac{\partial}{\partial \phi} - \sin \phi \csc \theta \frac{\partial}{\partial \theta}, \\ \partial_3 &= \frac{\partial}{\partial \phi}. \end{aligned} \quad (49)$$

The prolate case ($e_2 = e_3$) is obtained in the limit $k_1 = 0$ and $k_2 = 1$. Proceeding in a similar way to the oblate case, for the prolate rigid body we have that the suitable parameterization of the S^2 sphere is

$$u_1 = \cos \theta, \quad u_2 = \sin \theta \cos \phi, \quad u_3 = \sin \theta \sin \phi, \quad (50)$$

obtaining that the components of the gradient operator (43) become

$$\begin{aligned} \partial_1 &= \frac{\partial}{\partial \theta}, \\ \partial_2 &= -\cos \theta \cot \phi \frac{\partial}{\partial \theta} + \sin \theta \csc \phi \frac{\partial}{\partial \phi}, \\ \partial_3 &= -\sin \theta \cot \theta \frac{\partial}{\partial \theta} + \cos \theta \csc \phi \frac{\partial}{\partial \phi}. \end{aligned} \quad (51)$$

VII. CLASSIFICATION OF THE $SL(2, \mathbb{R})$ GAUGE TRANSFORMATIONS

We are now in position to fully classify the $SL(2, \mathbb{R})$ gauge transformations of the Eulerian top. The transformations are divided into three general sets where each set belongs to a different fixed form of the $SL(2, \mathbb{R})$ matrix. As we will discuss, for some sets there are different cases where the Hamiltonian vector fields belong to inequivalent forms of the three dimensional Lie algebras. Our analysis follows closely to the one performed in [23]. Along our discussion we use both the nomenclature commonly used in the Lie algebras literature (see for instance [25]) and the one introduced by Bianchi in his classification of three dimensional Lie algebra [15].

A. The Bianchi IX or $SO(3)$ algebra: $c = 0$ and $d \neq 0$

A generic $SL(2, \mathbb{R})$ group element in this case has the form

$$g = \begin{pmatrix} a & b \\ 0 & 1/a \end{pmatrix} \quad \text{with } b \in \mathbb{R}. \quad (52)$$

Without lossing generality we can assume that $a > 0$. These group elements are of the type AN in the Iwasawa decomposition of the special linear group [31], and as a generic characteristic they do not change the type of group orbit. In this case, we can define $Y_{u_1} \equiv aX_{u_1}$, $Y_{u_2} \equiv aX_{u_2}$ and $Y_{u_3} \equiv aX_{u_3}$ and we obtain the $SO(3)$ Lie algebra

$$[Y_{u_1}, Y_{u_2}] = Y_{u_3}, \quad [Y_{u_2}, Y_{u_3}] = Y_{u_1}, \quad [Y_{u_3}, Y_{u_1}] = Y_{u_2}. \quad (53)$$

According to equation (39) the transformed $SL(2, \mathbb{R})$ dynamical quantities are

$$H = \frac{1}{2} [(ae_1 + b)u_1^2 + (ae_2 + b)u_2^2 + (ae_3 + b)u_3^2] - \frac{1}{2}(ae_0 + b) = 0, \quad \text{and} \quad N = \frac{1}{2a} [u_1^2 + u_2^2 + u_3^2] - \frac{1}{2a} = 0, \quad (54)$$

whereas the Hamiltonian generator is given by

$$Y_G = -(ae_1 + b)u_1Y_{u_1} - (ae_2 + b)u_2Y_{u_2} - (ae_3 + b)u_3Y_{u_3}. \quad (55)$$

Notice that the Casimir N is indeed the same as the Casimir l (equation (38)), and therefore geometrically it is represented by an S^2 sphere. On the contrary the geometry of the Casimir H and the expression of the Hamiltonian generator Y_G depends on the values of the coefficients $ae_i + b$. Determining these quantities requires an analysis by cases. The one corresponding to $b = 0$ is the simplest case and describes the evolution of the dimensionless angular momentum of an asymmetric rigid body, because the Casimir H is given simply by h . According to the Euler equation (40) the physics occurs in the intersection of the surfaces l and h , and for $b = 0$ in the asymmetric situation this occurs in the 10 different forms listed in tables II and III, as well as with its 3+3 symmetric limits.

Exceptional cases take place if e_0 has some of the three values e_1 , e_2 or e_3 . If $e_0 = e_1$ (or $e_0 = e_3$), the hyperboloid and the sphere are tangent in only two points: $(u_1, u_2, u_3) = (\pm 1, 0, 0)$ (or $(u_1, u_2, u_3) = (0, 0, \pm 1)$). But if $e_0 = e_2$ the hyperboloid and the sphere are tangent in two big circles forming the separatrix, with two opposite unstable points at $(u_1, u_2, u_3) = (0, \pm 1, 0)$.

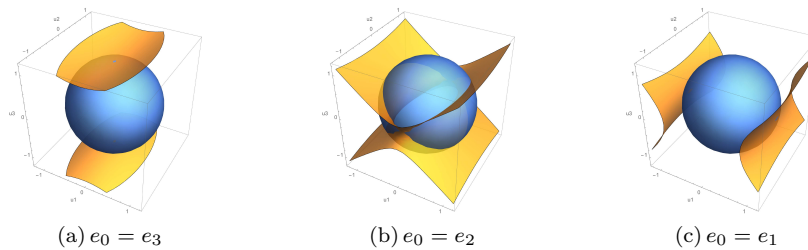


FIG. 2: Intersections of the Casimir function (21) and the unitary sphere when $e_0 = e_i$, $i = 1, 2, 3$.

For $b \neq 0$ the Casimir H does not coincide anymore with h and its geometries are discussed next.

1. Asymmetric cases: $D_2 \subset SO(3)$

As discussed in [24], for the asymmetric case, the Dihedral D_2 group is a subgroup of $SO(3)$. We use the notation $D_2 \subset SO(3)$ to emphasize this property. In order to determine the geometries of the Casimir function H it is important to know the relation between the energy parameter e_0 and the inertia parameter e_2 . We have as in section V, two different cases.

- Case $e_2 < e_0 < e_1$

In this first case, the dimensionless energy parameter e_0 lies between the values of e_1 and e_2 . To obtain the different geometries of the H surface, we allow the free parameter b to take every value in the real line. This gives origin to 9 different generic cases. For instance, if $-\infty < b < -ae_1$, the three coefficients $ae_i + b$ are negative (see table VI) as well as the quantity $ae_0 + b$. The resulting surface is an ellipsoid whose larger axis is in the direction u_1 , whereas the shorter axis is in direction u_3 . In the limit situation $b \rightarrow -\infty$ the H surface degenerates into an S^2 sphere. When the parameter takes the exact value $b = -ae_1$ the H surface becomes an elliptic cylinder with symmetry axis in the direction u_1 . If now the values of the parameter are in the interval $-ae_1 < b < -ae_0$, the H surface becomes an elliptic hyperboloid of one sheet around axis u_1 . The different surfaces H obtained for the different values of the parameter b are summarized in table VI.

$ae_1 + b$	$ae_0 + b$	$ae_2 + b$	$ae_3 + b$	H surface
< 0	< 0	< 0	< 0	elliptic ellipsoid with larger axis in u_1 and shorter axis in u_3
$= 0$	< 0	< 0	< 0	elliptic cylinder around u_1
> 0	< 0	< 0	< 0	elliptic hyperboloid of one sheet around u_1
> 0	$= 0$	< 0	< 0	elliptic cone around u_1
> 0	> 0	< 0	< 0	elliptic hyperboloid of two sheets around u_1
> 0	> 0	$= 0$	< 0	hyperbolic cylinder around u_2
> 0	> 0	> 0	< 0	elliptic hyperboloid of one sheet around u_3
> 0	> 0	> 0	$= 0$	elliptic cylinder around u_3
> 0	> 0	> 0	> 0	elliptic ellipsoid with larger axis in u_3 and shorter axis in u_1

TABLE VI: Classification of the nine different geometries for the Casimir function H (54), in the asymmetric cases satisfying $e_2 < e_0 < e_1$. For each of these cases the Lie algebra of the Hamiltonian vector fields associated to the coordinates u_i 's is of Bianchi IX type.

It is important to stress that independently of the geometry of H the important fact is that the intersection of the surfaces H and N is the same. In other words, the dynamics of the asymmetric rigid body, which takes place on the intersection of the two Casimir surfaces according to the Euler equations (40), is invariant under a gauge transformation of the form (52), and the geometrical meaning of the transformation consists in changing the geometry of the Casimir H .

- Case $e_3 < e_0 < e_2$

This case is very similar to the previous one, but this time, because e_0 belongs to a different interval, the intersections of the Casimir functions H and N occur in different regions. In the previous case the intersections occur in the upper region respect to the separatrix, and are given by elliptical curves on the sphere around the axis u_3 , whereas in this case the intersections occur below the separatrix and are given by elliptical curves on the sphere around the axis u_1 . In table VII we summarize the geometries for the different possible values of the parameter b .

Finally notice that in transformations (52) we have considered that $a > 0$. If instead we consider group elements such that $a < 0$, the gauge transformation changes the order of the inertia parameters from $e_3 < e_2 < e_1$ to $e_1 < e_2 < e_3$ and therefore to an interchange in the solutions of the modulus of the elliptic functions $k_1^2 \leftrightarrow k_2^2$ (see table I).

2. Symmetric cases: $SO(2) \subset SO(3)$

As limiting cases of the Bianchi IX asymmetric rigid bodies we obtain either the oblate (prolate) Bianchi IX symmetric rigid bodies. For these cases $SO(2)$ is a subgroup of the $SO(3)$ Lie algebra (53) and using the notation of [24] we will denote both cases as $SO(2) \subset SO(3)$. As discussed in section VI, the symmetric cases are obtained by setting $e_1 = e_2$ (oblate) and $e_2 = e_3$ (prolate). Taking the oblate limit in table VII we obtain seven different geometries for the surface H which are summarized in the following table.

$ae_1 + b$	$ae_2 + b$	$ae_0 + b$	$ae_3 + b$	H Surface
< 0	< 0	< 0	< 0	elliptic ellipsoid with larger axis in u_1 and shorter axis in u_3
$= 0$	< 0	< 0	< 0	elliptic cylinder around u_1
> 0	< 0	< 0	< 0	elliptic hyperboloid of one sheet around u_1
> 0	$= 0$	< 0	< 0	hyperbolic cylinder around u_2
> 0	> 0	< 0	< 0	elliptic hyperboloid of two sheets around u_3
> 0	> 0	$= 0$	< 0	elliptic cone around u_3
> 0	> 0	> 0	< 0	elliptic hyperboloid of one sheet around u_3
> 0	> 0	> 0	$= 0$	elliptic cylinder around u_3
> 0	> 0	> 0	> 0	elliptic ellipsoid with larger axis in u_3 and shorter axis in u_1

TABLE VII: Classification of the nine different geometries for the Casimir function H (54), in the asymmetric cases satisfying $e_3 < e_0 < e_2$. For each of these cases the Lie algebra of the Hamiltonian vector fields associated to the coordinates u_i 's is of Bianchi IX type.

$ae_1 + b$	$ae_0 + b$	$ae_3 + b$	H Surface
< 0	< 0	< 0	oblate ellipsoid with larger axis in u_1
$= 0$	< 0	< 0	parallel $u_1 u_2$ planes
> 0	< 0	< 0	circular hyperboloid of two sheets around u_3
> 0	$= 0$	< 0	two sheet circular cone around u_3
> 0	> 0	< 0	circular hyperboloid of one sheet around u_3
> 0	> 0	$= 0$	circular cylinder around u_3
> 0	> 0	> 0	oblate ellipsoid with larger axis in u_3

TABLE VIII: Classification of the seven different geometries for the Casimir function H (54), in the oblate symmetric case. For each of these cases the Lie algebra of the Hamiltonian vector fields associated to the coordinates u_i 's is of Bianchi IX type.

In a similar way, the prolate symmetric cases are obtained as limiting cases of the classification in table VI. There are again seven different geometries and they differ respect to the ones in the oblate case in the orientation of the axial axis of symmetry. Because to obtain them is straightforward we do not go into details.

B. Cases: $c \neq 0$ and $d = 0$

A generic $SL(2, \mathbb{R})$ group element of this case has the form

$$g = \begin{pmatrix} a & b \\ -1/b & 0 \end{pmatrix} \quad \text{with } b \in \mathbb{R} \setminus \{0\}. \quad (56)$$

The case $a = 0$ only interchanges the role of the Casimir functions h and l and therefore is of Bianchi type IX. Here without loosing generality we assume that $a > 0$. In this case the Lie algebra (47) reduces to the form

$$[X_{u_1}, X_{u_2}] = -\frac{1}{b}e_3X_{u_3}, \quad [X_{u_2}, X_{u_3}] = -\frac{1}{b}e_1X_{u_1}, \quad [X_{u_3}, X_{u_1}] = -\frac{1}{b}e_2X_{u_2}. \quad (57)$$

The Casimir function H has the same form as (54) and therefore its geometries are given according to tables VI and VII, whereas the Casimir function N in this case is proportional to h : $N = -h/b$, and therefore its geometries are classified according to tables II and III. As we discuss next the group elements (56) act as gauge transformations in two inequivalent Lie algebras. The first one corresponds to the Bianchi VIII or $SO(2, 1)$ Lie algebra whereas the second one to the Bianchi VI₀ or $ISO(1, 1)$ algebra. The difference of these two algebras is due to the value of the e_2 inertia parameter. If $e_2 \neq 0$ we are in the Bianchi VIII case and if $e_2 = 0$ we have the Bianchi VI₀ case. We stress that the generators in both algebras describe in general an asymmetric body. Strictly speaking the system does not corresponds to a rigid body anymore because the Casimir function N is not longer S^2 . Sometimes in the literature these systems are called generalized rigid bodies [10]. Finally we comment that as in the previous subsection, taking $a < 0$ in (56) produces a different order in the inertia parameters.

1. Bianchi VIII cases

The Lie algebra (57) is in the Bianchi VIII class if $e_2 \neq 0$ as we can easily see. First remember that $e_1 > 0$, $e_3 = -|e_3| < 0$ and rewrite $e_2 \equiv \epsilon|e_2|$ where $\epsilon \equiv \text{sign}(e_2)$ since e_2 can be either positive or negative. If we redefine the

generators (46) in the way

$$X_{u_1} \equiv -\frac{1}{b}\sqrt{|e_2||e_3|}Y_{u_1}, \quad X_{u_2} \equiv -\frac{1}{b}\sqrt{e_1|e_3|}Y_{u_2}, \quad X_{u_3} \equiv -\frac{1}{b}\sqrt{e_1|e_2|}Y_{u_3}, \quad (58)$$

we obtain the $SO(2,1)$ Lie algebra

$$[Y_{u_1}, Y_{u_2}] = -Y_{u_3}, \quad [Y_{u_2}, Y_{u_3}] = +Y_{u_1}, \quad [Y_{u_3}, Y_{u_1}] = \epsilon Y_{u_2}. \quad (59)$$

The symmetric limits can be taken directly in expressions (58) by setting $e_1 = e_2$ (oblate) and $e_2 = e_3$ (prolate).

2. Bianchi VI₀ case

The second inequivalent Lie algebra emerges when $e_2 = 0 \Rightarrow e_1 = -e_3$. In this case the geometry of the Casimir function N corresponds to a parabolic cylinder, according to tables II and III. Redefining the generators (46) as $X_{u_i} \equiv -\sqrt{e_1}Y_{u_i}/b$, for $i = 1, 3$ and $X_{u_2} \equiv -e_1Y_{u_2}/b$, the Lie algebra is of Bianchi VI₀ type

$$[Y_{u_1}, Y_{u_2}] = -Y_{u_3}, \quad [Y_{u_2}, Y_{u_3}] = +Y_{u_1}, \quad [Y_{u_3}, Y_{u_1}] = 0. \quad (60)$$

Notice in this case the expression of the vector fields (46) reduce to

$$X_{u_1} = -u_3\partial_2, \quad X_{u_2} = u_1\partial_3 + u_3\partial_1, \quad X_{u_3} = -u_1\partial_2, \quad (61)$$

where the value of the modulus in the Jacobi elliptic functions in the partials (43) are $k_1^2 = k_2^2 = 1/2$. There is not symmetric limit for this case.

C. Case $c \neq 0, d \neq 0$

In this case both Casimir functions are given by linear combinations of h and l . H is given by expression (54) whereas N is given by a similar expression

$$N = \frac{1}{2}[(ce_1 + d)u_1^2 + (ce_2 + d)u_2^2 + (ce_3 + d)u_3^2] - \frac{1}{2}(ce_0 + d) = 0. \quad (62)$$

For these gauge transformations the Lie algebra (47) can be rewritten as

$$[Y_{u_1}, Y_{u_2}] = \epsilon_3 Y_{u_3}, \quad [Y_{u_2}, Y_{u_3}] = \epsilon_1 Y_{u_1}, \quad [Y_{u_3}, Y_{u_1}] = \epsilon_2 Y_{u_2}. \quad (63)$$

It turns out that the general asymmetric case contains seven cases which belong to four different type A algebras in the Bianchi classification, namely Bianchi IX, VIII, VII₀ and VI₀. Without losing generality we assume that $c > 0$. In table IX we summarize all the different cases, which we analyze below.

Situation	$ce_1 + d$	$ce_2 + d$	$ce_3 + d$	ϵ_1	ϵ_2	ϵ_3	Bianchi type
1	< 0	< 0	< 0	-1	-1	-1	IX
2	$= 0$	< 0	< 0	0	-1	-1	VII ₀
3	> 0	< 0	< 0	+1	-1	-1	VIII
4	> 0	$= 0$	< 0	+1	0	-1	VI ₀
5	> 0	> 0	< 0	+1	+1	-1	VIII
6	> 0	> 0	$= 0$	+1	+1	0	VII ₀
7	> 0	> 0	> 0	+1	+1	+1	IX

TABLE IX: Inequivalent Lie algebras for $SL(2\mathbb{R})$ gauge transformations with $c \neq 0$ and $d \neq 0$. All the algebras are of type A in the Bianchi classification, and describe an asymmetric rigid body.

1. Asymmetric Bianchi IX and VIII cases

Proceeding in a similar way to the analysis performed in section VII B, we write the three coefficients $ce_i + d \equiv \epsilon_i|ce_i + d| \neq 0$, where $\epsilon_i = \text{sign}(ce_i + d)$. As a consequence we define the generators Y_i 's through the relations

$$X_{u_1} \equiv \sqrt{|ce_3 + d||ce_2 + d|}Y_{u_1}, \quad X_{u_2} \equiv \sqrt{|ce_3 + d||ce_1 + d|}Y_{u_2}, \quad X_{u_3} \equiv \sqrt{|ce_2 + d||ce_1 + d|}Y_{u_3}, \quad (64)$$

which drive directly to equation (63). Situations 1, 3, 5 and 7 in table IX are of this form.

2. Asymmetric Bianchi VII₀ and VI₀ cases

- Situation 2: $d = -ce_1$, (Bianchi type VII₀ or $ISO(2)$).

According to table IX, this case is obtained when the coefficients of the $SL(2, \mathbb{R})$ transformation satisfy the relation $d = -ce_1$. In this case the Lie algebra (63) is of Bianchi type VII₀ or $ISO(2)$. Without lossing generality we can assume that $c > 0$ and the suitable redefinition of generators is given by

$$X_{u_1} \equiv c\sqrt{(e_1 - e_3)(e_1 - e_2)}Y_{u_1}, \quad X_{u_2} \equiv c\alpha Y_{u_2}, \quad X_{u_3} \equiv c\alpha\sqrt{\frac{e_1 - e_2}{e_1 - e_3}}Y_{u_3}. \quad (65)$$

Notice that additionally to the constant c , the generators X_{u_2} and X_{u_3} can be multiplied by an arbitrary constant α .

- Situation 4: $d = -ce_2 \neq 0$ (Bianchi VI₀ or $ISO(1, 1)$).

In this case the suitable redefinition of generators is given by

$$X_{u_1} \equiv c\alpha\sqrt{\frac{e_2 - e_3}{e_1 - e_2}}Y_1, \quad X_{u_2} \equiv c\sqrt{(e_1 - e_2)(e_2 - e_3)}Y_2, \quad X_{u_3} \equiv c\alpha Y_3. \quad (66)$$

- Situation 6: $d = -ce_3$ (Bianchi VII₀ or $ISO(2)$).

In this case the edefinition of generators is given as

$$X_{u_1} \equiv c\alpha Y_1, \quad X_{u_2} \equiv c\alpha\sqrt{\frac{e_1 - e_3}{e_2 - e_3}}Y_2, \quad X_{u_3} \equiv c\sqrt{(e_1 - e_3)(e_2 - e_3)}Y_3. \quad (67)$$

3. Oblate symmetric limits ($e_1 = e_2$)

Regarding the oblate symmetric limits ($e_1 = e_2$), we can see from table IX that they can be naturally obtained in the situations 1, 5, 6, 7, because in this case the signs of both $ce_1 + d$ and $ce_2 + d$ are the same. The cases 2, 3 and 4 all degenerate into one single case which is characterized by having $ce_1 + d = ce_2 + d = 0$ and whose algebra is of Bianchi II type and commonly known as the Heisenberg algebra Heis_3 [25]

Situation	$ce_1 + d$	$ce_3 + d$	Bianchi type	X_{u_1}	X_{u_2}	X_{u_3}
1	< 0	< 0	IX	$\sqrt{ ce_3 + d ce_1 + d }Y_{u_1}$	$\sqrt{ ce_3 + d ce_1 + d }Y_{u_2}$	$ ce_1 + d Y_{u_3}$
8	$= 0$	< 0	II	Y_{u_1}	Y_{u_2}	$3 d Y_{u_3}/2$
5	> 0	< 0	VIII	$\sqrt{ ce_3 + d ce_1 + d }Y_{u_1}$	$\sqrt{ ce_3 + d ce_1 + d }Y_{u_2}$	$ ce_1 + d Y_{u_3}$
6	> 0	$= 0$	VII ₀	Y_{u_1}	Y_{u_2}	$ ce_1 + d Y_{u_3}$
7	> 0	> 0	IX	$\sqrt{ ce_3 + d ce_1 + d }Y_{u_1}$	$\sqrt{ ce_3 + d ce_1 + d }Y_{u_2}$	$ ce_1 + d Y_{u_3}$

TABLE X: Oblate symmetric cases ($e_1 = e_2$) and the classification of the Lie-Poisson algebra of the Hamiltonian vector fields associated with the coordinates u_i .

4. Prolate symmetric limits ($e_2 = e_3$)

A similar result is obtained for the prolate limit ($e_2 = e_3$). For completeness we include the possibilities in the following table, but we do not going into details as the algebras resulting from taking this limit are the same as in the $e_1 = e_2$ case.

VIII. CONCLUSIONS

In this contribution we have fully classified the $SL(2, \mathbb{R})$ gauge transformations in the rigid body, using the generalized dynamics of Nambu. These gauge transformations are geometrical transformations of the Casimir functions that keep invariant its geometrical intersections once the Hamiltonian vector fields associated to the coordinates (the

Situation	$ce_1 + d$	$ce_3 + d$	Bianchi type
1	< 0	< 0	IX
2	$= 0$	< 0	VII ₀
3	> 0	< 0	VIII
8	> 0	$= 0$	II
7	> 0	> 0	IX

TABLE XI: Prolate symmetric cases ($e_2 = e_3$) and the classification of the Lie-Poisson algebra of the Hamiltonian vector fields associated with the coordinates u_i .

components of angular momentum in this case) satisfy a specific three dimensional Lie algebra. Some general results were previously reported in the literature [8, 10], and here we have constructed explicitly all the inequivalent Bianchi type A Lie algebras that are possible for this system. In particular we have worked out and classified the Hamiltonian vector fields for all the Lie algebras, something that has not been fully addressed in the literature. Our analysis is classical and can be considered as a first step in the direction to generalize the quantum results obtained for the symmetrical rigid bodies [10], to the asymmetrical ones. In fact there are some recent results in the study of the quantum spectrum of the rotational motion of asymmetric molecules [14, 16–19], that could be generalized to the extended rigid body problem, and could bring some light in the quantization of the system using the Nambu brackets for the asymmetric cases.

Although the rigid body problem has been extensively studied over the years, from the classical mechanics point of view there are still some interesting issues that could be addressed. For mentioning some of them, it would be very interesting to establish the nature of the generating function of the canonical transformations from the coordinates (32) used here and the angle-action coordinates for the extended rigid bodies. A similar study was recently performed for the simple pendulum [32], and because the dynamics of the rigid body reduces to the one of the simple pendulum when the Lie algebra is of Bianchi type VII₀ or $ISO(2)$, it must be possible to perform such an analysis. Another possibility is to study the $SL(3, \mathbb{R})$ canonical transformations of the rigid body in the context of the generalized bi-Hamiltonian dynamics and determine which canonical transformations generate the different ordering in the inertia parameters, because as we have discussed in this contribution, the $SL(2, \mathbb{R})$ gauge transformations give account only of the ones that interchange the parameters $e_1 \leftrightarrow e_3$. This analysis should be possible because the solutions for the asymmetric cases are given in terms of Jacobi elliptical functions and they incorporate in a natural way the transformations to generate the six different case of table I. For a recent review in the case of the simple pendulum see [33]. A further very interesting analysis is find the relation between the different gauge transformations studied in this paper with the role these same transformations play in the context of the generalized Hamilton-Jacobi theory of Nambu Mechanics recently studied in [34].

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